Series expansions for the third incomplete elliptic integral via partial fraction decompositions

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Abstract. We find convergent double series expansions for Legendre's third incomplete elliptic integral valid in overlapping subdomains of the unit square. Truncated expansions provide asymptotic approximations in the neighbourhood of the logarithmic singularity $(1,1)$ if one of the variables approaches this point faster than the other. Each approximation is accompanied by an error bound. For a curve with an arbitrary slope at $(1,1)$ our expansions can be rearranged into asymptotic expansions depending on a point on the curve. For reader's convenience we give some numeric examples and explicit expressions for low-order approximations.

Keywords: *Incomplete elliptic integral, series expansion, asymptotic approximation, partial fraction decomposition*

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1. Introduction. Legendre's incomplete elliptic integral (EI) of the third kind is defined by [5, formula 13.6(3)]

$$\Pi(\lambda,\nu,k)=\int\limits_0^{\lambda}\frac{dt}{(1+\nu t^2)\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{1}$$

It is one of the three canonical forms given by Legendre in terms of which all elliptic integrals can be expressed. We will only consider the most important case $0\le k\le 1$, $0\le\lambda\le 1$ and $\nu>-1$. Two other canonical forms can be expressed in terms of $\Pi(\lambda,\nu,k)$: setting $\nu=0$ gives the first incomplete elliptic integral, while setting $\nu=-k^2$ gives the second plus an elementary function (see [5, formulas 13.6(11)-13.6(13)]). When $\lambda=1$ the rhs of (1) is called the third complete elliptic integral.

Series expansions, asymptotic approximations and inequalities for the third incomplete elliptic integral have been studied by many authors. Major contributions were made by Radon [8], Carlson [1, 2], Carlson and Gustafson [3] and Lopez [6, 7]. Carlson showed in [1] that $\Pi(\lambda,\nu,k)$ can be expressed in terms of Lauricella hypergeometric function F_D of three variables (see [4]). In the same paper he noted that one can derive rapidly convergent expansions by first expressing Legendre's incomplete EIs in a different form. This form later became known as symmetric standard EIs. Today, most authors consider asymptotic approximations for these symmetric EIs. Since Legedre's incomplete elliptic integrals are connected with symmetric EIs by certain simple relations (see [1, 2]), it is possible to reformulate expansions presented here in terms of symmetric elliptic integrals.

In this paper we give convergent double series expansion for the third incomplete elliptic integral (1) valid in two overlapping subregions of the unit square $[0,1]\times[0,1]$ in (k^2,λ^2)-plane defined by

$$\text{Region I}=\{k^2,\lambda^2\in[0,1]:\lambda^2<1/(2-k^2)\Leftrightarrow(1-k^2)\lambda^2/(1-\lambda^2)<1\}, \tag{2}$$

$$\text{Region II}=\{k^2,\lambda^2\in[0,1]:\lambda^2>2-1/k^2\Leftrightarrow(1-\lambda^2)k^2/(1-k^2)<1\}. \tag{3}$$

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Combined these subregions cover the unit square completely. Truncating our expansions one gets asymptotic approximations for $(1-\lambda)/(1-k) \to 0$ or $(1-\lambda)/(1-k) \to \infty$, i.e. for curves with endpoint on the top or right side of the unit square or with endpoint $\lambda = k = 1$ and horizontal or vertical tangent at that point. In all cases we give explicit bounds for the remainder term.

For a given smooth curve approaching the singular point $(1,1)$ from any direction our double series expansions can be rearranged into asymptotic expansions, where a point on the curve plays the role of an asymptotic parameter. In the last section of the paper we illustrate our results by numerical examples. Rather good precision (with less than one percent relative error) is reached in most cases already by the first asymptotic term. For reader's convenience we also give explicit expressions for first several terms of each expansion.

2. Partial fraction decompositions. For a non-negative integer n and $\alpha = 0,1$ define

$$\phi_n^\alpha(t) = (1-t)^{-n} + (-1)^\alpha(1+t)^{-n}, \quad \psi_n^\alpha(t) = \begin{cases} [(1-t)^{-n} - (-1)^\alpha(1+t)^{-n} - 2\alpha]/n, & n > 0, \\ (-1)^\alpha \ln(1+t) - \ln(1-t), & n = 0. \end{cases} \tag{4}$$

The second definition is consistent in the sense that $\lim\limits_{n\to 0} \psi_n^\alpha(t) = (-1)^\alpha \ln(1+t) - \ln(1-t)$, $\alpha = 0,1$. Our first lemma is straightforward.

Lemma 1 *The functions ϕ_n^α and ψ_n^α are related by:*

$$\int\limits_0^\lambda \phi_n^\alpha(t)dt = \psi_{n-1}^\alpha(\lambda), \qquad n = 1,2,\ldots, \qquad \alpha = 0,1. \tag{5}$$

Lemma 2 *Let $n < 2j$ be non-negative integers. Then the following partial fraction decomposition holds true:*

$$\frac{t^n}{(1-t^2)^j} = \beta_j^n[1]\phi_1^\alpha(t) + \beta_j^n[2]\phi_2^\alpha(t) + \cdots + \beta_j^n[j]\phi_j^\alpha(t), \tag{6}$$

where $\alpha = n \bmod 2$, $\beta_j^n[j] = 2^{-j}$ and the numbers $\beta_j^n[j-k]$, $k = 1,2,\ldots,j-1$, are found from the following recurrence (here $\binom{n}{k} = 0$ when $k > n$):

$$\beta_j^n[j-k] = \frac{(-1)^k}{2^j}\binom{n}{k} - \sum_{i=0}^{k-1}(-2)^{i-k}\binom{j}{k-i}\beta_j^n[j-i], \qquad k = 1,2,\ldots,j-1, \tag{7}$$

or explicitly from

$$\beta_j^n[j-k] = 2^{-j-k}\sum_{i=0}^{k}(-2)^i\binom{n}{i}\binom{k+j-1-i}{j-1}, \quad 0 \le k \le j. \tag{8}$$

Proof. For computing the coefficients of the partial fraction decomposition we use the trick routinely applied for calculating residues. Writing

$$\frac{t^n}{(1-t)^j(1+t)^j} = \frac{\beta_j^n[1]}{1-t} + \frac{\beta_j^n[2]}{(1-t)^2} + \cdots + \frac{\beta_j^n[j]}{(1-t)^j} + (-1)^n\frac{R(t)}{(1+t)^j},$$

where

$$\frac{R(t)}{(1+t)^j} = \frac{\beta_j^n[1]}{1+t} + \frac{\beta_j^n[2]}{(1+t)^2} + \cdots + \frac{\beta_j^n[j]}{(1+t)^j},$$

and multiplying throughout by $(1-t^2)^j$, we get:

$$t^n = (1+t)^j(\beta_j^n[j] + \beta_j^n[j-1](1-t) + \cdots + \beta_j^n[1](1-t)^{j-1}) + (1-t)^jR(t).$$

Setting $t = 1$ immediately reveals $\beta_j^n[j] = 2^{-j}$. Differentiate k times and set $t = 1$:

$$[t^n]_{|t=1}^{(k)} = \sum_{i=0}^{k}\binom{k}{i}[(1+t)^j]_{|t=1}^{(k-i)}\beta_j^n[j-i](-1)^i i!, \qquad k = 1,2,\ldots,j-1.$$

We have:

$$[t^n]^{(k)}_{|t=1} = \begin{cases} n!/(n-k)!, & k \le n \\ 0, & k > n \end{cases}, \quad [(1+t)^j]^{(k-i)}_{|t=1} = \frac{j!}{(j-k+i)!}2^{j-k+i}.$$

Thus the numbers $\beta^n_j[j-k]$ can be found via the following recurrence (starting with $\beta^n_j[j] = 2^{-j}$ and counting downwards):

$$\beta^n_j[j-k](-1)^k k! 2^j = \frac{n!}{(n-k)!} - \sum_{i=0}^{k-1} \binom{k}{i} \frac{j!}{(j-k+i)!} 2^{j-k+i} \beta^n_j[j-i](-1)^i i!, \qquad k = 1, 2, \dots, j-1.$$

A simple rearrangement of this formula yields (7) which can also be written as

$$\sum_{i=0}^{k} (-2)^{i-k} \binom{j}{k-i} \beta^n_j[j-i] = \frac{(-1)^k}{2^j}\binom{n}{k}.$$

This recurrence can be solved for $\beta^n_j[j-i]$ using the inversion formula (see [9])

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{m}{n-k} d_k \quad \Rightarrow \quad d_n = \sum_{k=0}^{n} (-1)^k \binom{n+m-1-k}{n-k} a_k,$$

which leads to (8). $\square$

In the sequel we will only need even n therefore we put $\phi_n(t) = \phi^0_n(t)$ and $\psi_n(t) = \psi^0_n(t)$ to simplify notation.

Lemma 3 *The following partial fraction decomposition holds true for* $j = 0, 1, \dots$:

$$\frac{t^{2j}}{(1+\nu t^2)(1-t^2)^{j+1}} = \frac{(-1)^j \nu}{(1+\nu)^{j+1}(1+\nu t^2)} + \sum_{k=0}^{j} \frac{(-1)^k \phi_{j+1-k}(t)}{(1+\nu)^{k+1}} \sum_{i=0}^{k} a_{j+1-k,i}(1+\nu)^i, \tag{9}$$

where the numbers $a_{n,m}$ *are expressed in terms of the numbers* $\beta^k_p[i]$ *defined by* (7) *or* (8) *by*

$$a_{n,m} = (-1)^m \beta^{2n+2m-4}_{n+m}[n], \quad n = 1, \dots, j+1; \quad m = \begin{cases} 1, \dots, j, & n = 1 \\ 0, \dots, j+1-n, & n > 1 \end{cases}; \quad a_{1,0} = \frac{1}{2}. \tag{10}$$

Remark 1. This lemma contains two statements: one is formula (10) for the coefficients of expansion (9), and the other is that the numbers $a_{n,m}$ *are independent of* j. That is when j increases we do not need to update all the numbers $a_{n,m}$, instead we keep all previously calculated numbers and complement them with new $j+1$ numbers with indices summing up to $j+1$: $n+m = j+1$.

Proof. We will use induction in j. For $j = 0$ we can verify directly that:

$$\frac{1}{(1+\nu t^2)(1-t^2)} = \frac{\nu}{(1+\nu)(1+\nu t^2)} + \frac{1}{2(1+\nu)}\left[\frac{1}{1-t} + \frac{1}{1+t}\right].$$

Suppose now that (9) holds for a fixed j. Then for $j+1$,

$$\frac{t^{2j+2}}{(1+\nu t^2)(1-t^2)^{j+2}} = \frac{t^{2j}}{(1+\nu)(1-t^2)^{j+2}} - \frac{t^{2j}}{(1+\nu)(1+\nu t^2)(1-t^2)^{j+1}}.$$

Now using (6) with $n = 2j$, $m = j+2$ for the first term and (9) for the second gives after collecting coefficients at $\phi_i(t)$:

$$\begin{aligned}
\frac{t^{2j+2}}{(1+\nu t^2)(1-t^2)^{j+2}} &= \frac{(-1)^{j+1}\nu}{(1+\nu)^{j+2}(1+\nu t^2)} + \\
&+ \frac{(-1)^{j+1}\left(a_{1,0} + a_{1,1}(1+\nu) + \dots + a_{1,j}(1+\nu)^j\right) + \beta^{2j}_{j+2}[1](1+\nu)^{j+1}}{(1+\nu)^{j+2}}\phi_1(t) + \\
&+ \frac{(-1)^{j}\left(a_{2,0} + a_{2,1}(1+\nu) + \dots + a_{2,j-1}(1+\nu)^{j-1}\right) + \beta^{2j}_{j+2}[2](1+\nu)^{j-1}}{(1+\nu)^{j}}\phi_2(t) + \\
&+ \dots + \frac{(-1)a_{j+1,0} + \beta^{2j}_{j+2}[j+1](1+\nu)}{(1+\nu)^2}\phi_{j+1}(t) + \frac{\beta^{2j}_{j+2}[j+2]}{1+\nu}\phi_{j+2}(t).
\end{aligned}$$

Now define

$$a_{1,j+1} = (-1)^{j+1}\beta_{j+2}^{2j}[1], \quad a_{2,j} = (-1)^{j}\beta_{j+2}^{2j}[2], \quad \ldots, \quad a_{j+2,0} = \beta_{j+2}^{2j}[j+2].$$

This shows that expansion for $j+1$ has the form (9) with numbers $a_{n,m}$ given by (10). $\square$

Formula (9) combined with Lemma 1 leads to the following evaluation

$$\int_0^\lambda \frac{t^{2j}\,dt}{(1+\nu t^2)(1-t^2)^{j+1}} = \frac{(-1)^j\sqrt{\nu}}{(1+\nu)^{j+1}}\arctan(\lambda\sqrt{\nu}) + \sum_{n=0}^{j} d_{j,n+1}(\nu)\psi_n(\lambda), \tag{11}$$

where we introduced the notation

$$d_{j,n+1}(\nu) = \frac{(-1)^{j-n}}{(1+\nu)^{j-n+1}}\sum_{i=0}^{j-n} a_{n+1,i}(1+\nu)^i, \quad n = 0,1,2,\ldots,j. \tag{12}$$

Here and henceforth $\sqrt{\nu}\arctan(\lambda\sqrt{\nu})$ will be understood as $(\sqrt{-\nu})\text{arctanh}(\lambda\sqrt{-\nu})$ for negative ν.

Another way of decomposing the left hand side of (9) is given in the next lemma.

Lemma 4 *The following expansions hold true for $j = 0,1,\ldots$:*

$$\frac{t^{2j}}{(1+\nu t^2)(1-t^2)^{j+1}} = \left[\frac{\nu}{1+\nu}\right]^{j+1}\frac{t^{2j}}{1+\nu t^2} + \sum_{n=1}^{j+1}\frac{\nu^{j-n+1}}{(1+\nu)^{j-n+2}}\frac{t^{2j}}{(1-t^2)^n}, \tag{13}$$

and

$$\frac{t^{2j}}{1+\nu t^2} = \frac{(-1/\nu)^j}{1+\nu t^2} - \sum_{i=0}^{j-1}(-1/\nu)^{j-i}t^{2i}. \tag{14}$$

Proof. Both expansions can be verified by using the summation formula for a finite geometric progression. $\square$

Euler's integral representation for the Gauss hypergeometric function ${}_2F_1$ (after simple variable change) combined with (5) and (6) gives

$$\int_0^\lambda \frac{t^{2j}\,dt}{(1-t^2)^n} = \frac{\lambda^{2j+1}}{2j+1}{}_2F_1(n, j+1/2; j+3/2; \lambda^2) = \sum_{i=1}^{n}\beta_n^{2j}[i]\psi_{i-1}(\lambda). \tag{15}$$

Lemma 4 and formula (15) provide two alternative ways to evaluate the integral in (11). We have

$$\int_0^\lambda \frac{t^{2j}\,dt}{(1+\nu t^2)(1-t^2)^{j+1}} = \left[\frac{\nu}{1+\nu}\right]^{j+1}\int_0^\lambda \frac{t^{2j}\,dt}{1+\nu t^2} + \frac{\lambda^{2j+1}}{2j+1}\sum_{n=1}^{j+1}\frac{\nu^{j-n+1}{}_2F_1(n, j+1/2; j+3/2; \lambda^2)}{(1+\nu)^{j-n+2}} \tag{16}$$

$$= \frac{(-1)^j\sqrt{\nu}}{(1+\nu)^{j+1}}\arctan(\lambda\sqrt{\nu}) + \frac{(-1)^j}{(1+\nu)^{j+1}}\sum_{i=0}^{j-1}(-\nu)^{i+1}\frac{\lambda^{2i+1}}{2i+1} + \sum_{n=1}^{j+1}\frac{\nu^{j-n+1}}{(1+\nu)^{j-n+2}}\sum_{i=1}^{n}\beta_n^{2j}[i]\psi_{i-1}(\lambda). \tag{17}$$

3. Expansions for the third incomplete elliptic integral. The following relation will be of great help:

$$\Pi(\lambda,\nu,k) = \Pi(\nu,k) - \frac{1}{(1+\nu)\sqrt{1-k^2}}\Pi\left(-\nu/(1+\nu), \sqrt{1-\lambda^2}, \sqrt{-k^2/(1-k^2)}\right), \tag{18}$$

where $\Pi(\nu,k)$ is the complete elliptic integral of the third kind. Here we choose the branch of the first square root that is positive for positive values of $1-\lambda^2$. The choice of the branch of the second square root is immaterial since Π depends on the squared second argument only. This relation can be easily verified by representing the integral over $(0,\lambda)$ from (1) as the difference of integrals over $(0,1)$ and $(\lambda,1)$ and introducing the new integration variable $u^2 = 1-t^2$.

Theorem 1 *Let $k, \lambda \in$ Region I and $\nu > -1$. Then for any positive integer N the following expansions hold true*

$$\Pi(\lambda,\nu,k) = \sum_{j=0}^{N-1}(-1)^j\frac{(1/2)_j}{j!}(1-k^2)^j\left[\frac{(-1)^j\sqrt{\nu}}{(1+\nu)^{j+1}}\arctan(\lambda\sqrt{\nu})+\sum_{n=0}^{j}d_{j,n+1}(\nu)\psi_n(\lambda)\right]+R_{1,N}(k,\lambda,\nu), \tag{19}$$

and

$$\Pi(\lambda,\nu,k) = \sum_{j=0}^{N-1}\frac{(1/2)_j}{j!}(1-k^2)^j\left\{\frac{\sqrt{\nu}}{(1+\nu)^{j+1}}\arctan(\lambda\sqrt{\nu})\right.$$

$$\left.+\frac{1}{(1+\nu)^{j+1}}\sum_{i=0}^{j-1}(-\nu)^{i+1}\frac{\lambda^{2i+1}}{2i+1}+\sum_{n=1}^{j+1}\frac{(-1)^j\nu^{j-n+1}}{(1+\nu)^{j-n+2}}\sum_{i=1}^{n}\beta_n^{2j}[i]\psi_{i-1}(\lambda)\right\}+R_{1,N}(k,\lambda,\nu), \tag{20}$$

where the functions $\psi_n(\lambda)$ and $d_{j,n}(\nu)$ are defined by (4) and (12), respectively, and the remainder term satisfies

$$|R_{1,N}(k,\lambda,\nu)| \le \frac{(1/2)_N\lambda}{2\min(1,1+\nu)NN!}\left[\frac{(1-k^2)\lambda^2}{1-\lambda^2}\right]^N. \tag{21}$$

Proof. Put $k'^2 = 1-k^2$ and calculate using binomial expansion and termwise integration:

$$\Pi(\lambda,\nu,k) = \int_0^\lambda \frac{dt}{(1+\nu t^2)\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^\lambda \frac{dt}{(1+\nu t^2)(1-t^2)}\left(1+\frac{k'^2t^2}{1-t^2}\right)^{-1/2}$$

$$= \int_0^\lambda \frac{dt}{(1+\nu t^2)(1-t^2)}\left(\sum_{j=0}^{N-1}(-1)^j\frac{(1/2)_j}{j!}\frac{k'^{2j}t^{2j}}{(1-t^2)^j}+\sum_{j=N}^{\infty}(-1)^j\frac{(1/2)_j}{j!}\frac{k'^{2j}t^{2j}}{(1-t^2)^j}\right)$$

$$= \sum_{j=0}^{N-1}(-1)^j\frac{(1/2)_j}{j!}k'^{2j}\int_0^\lambda\frac{t^{2j}dt}{(1+\nu t^2)(1-t^2)^{j+1}}+R_{1,N}(k,\lambda,\nu), \tag{22}$$

where the remainder term is given by

$$R_{1,N}(k,\lambda,\nu) = \sum_{j=N}^{\infty}(-1)^j\frac{(1/2)_j}{j!}(1-k^2)^j\int_0^\lambda\frac{t^{2j}dt}{(1+\nu t^2)(1-t^2)^{j+1}}. \tag{23}$$

Using representations (11) and (17) for the integral in (22) we get (19) and (20), respectively. To prove the error bound (21) we note that the series (23) has alternating signs. The following estimate shows that the terms in (23) monotonically decrease in absolute value:

$$\frac{(1/2)_{j+1}}{(j+1)!}(1-k^2)^{j+1}\int_0^\lambda\frac{t^{2j+2}dt}{(1+\nu t^2)(1-t^2)^{j+2}} = \frac{(1/2)_j(1/2+j)}{j!(j+1)}\int_0^\lambda\frac{(1-k^2)^jt^{2j}}{(1+\nu t^2)(1-t^2)^{j+1}}\frac{t^2(1-k^2)}{(1-t^2)}dt$$

$$\le \frac{(1/2)_j}{j!}(1-k^2)^j\int_0^\lambda\frac{t^{2j}}{(1+\nu t^2)(1-t^2)^{j+1}}\frac{\lambda^2(1-k^2)}{(1-\lambda^2)}dt \le \frac{(1/2)_j}{j!}(1-k^2)^j\int_0^\lambda\frac{t^{2j}dt}{(1+\nu t^2)(1-t^2)^{j+1}}.$$

The last inequality is due to (2). Hence we are in the position to apply the Leibnitz convergence test and the remainder term does not exceed the first term in (23):

$$|R_{1,N}(k,\lambda,\nu)| \le \frac{(1/2)_N}{N!}(1-k^2)^N\int_0^\lambda\frac{t^{2N}dt}{(1+\nu t^2)(1-t^2)^{N+1}} \le \frac{(1/2)_N(1-k^2)^N}{N!\min(1,1+\nu)}\int_0^\lambda\frac{t^{2N}dt}{(1-t^2)^{N+1}}.$$

The integral on the right hand side satisfies the following asymptotically precise (as $\lambda \to 1$) estimate

$$f_1(\lambda) \equiv \int_0^\lambda \frac{t^{2N} dt}{(1-t^2)^{N+1}} \leq \frac{\lambda^{2N+1}}{2N(1-\lambda^2)^N} \equiv f_2(\lambda), \tag{24}$$

which is valid for all $\lambda \in (0,1)$ and $N > 0$ (not necessarily integer). Indeed, $f_1(0) = f_2(0) = 0$ and

$$\frac{f_1'(\lambda)}{f_2'(\lambda)} = \frac{2N}{2N+1-\lambda^2} < 1, \quad \lambda \in (0,1).$$

This estimate immediately leads to (21). □

Remark 2. The error bound (21) shows that expansions (19) and (20) are asymptotic for $\lambda \to 1$ with either constant k or $k \to 1$ in a way that $(1-\lambda)/(1-k) \to 0$, so that we can approach the singular point $(1,1)$ along a curve in (k,λ) plane having infinite slope at $(1,1)$. These expansions can be transformed into asymptotic expansion for $k, \lambda \to 1$ along an arbitrary curve if an ordering can be introduced into the matrix of functions forming the expansion that converts this matrix into an asymptotic scale. Indeed, we have the matrix

$$\begin{array}{cccc}
\psi_0(\lambda) & & & \\
\downarrow & & & \\
\arctan(\lambda\sqrt{\nu}) & & & \\
\downarrow & & & \\
(1-k^2)\arctan(\lambda\sqrt{\nu}) & (1-k^2)\psi_0(\lambda) & \leftarrow (1-k^2)\psi_1(\lambda) & \\
\downarrow & \downarrow & \downarrow & \\
(1-k^2)^2\arctan(\lambda\sqrt{\nu}) & (1-k^2)^2\psi_0(\lambda) & \leftarrow (1-k^2)^2\psi_1(\lambda) & \leftarrow (1-k^2)^2\psi_2(\lambda) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\cdots & \cdots & \cdots & \cdots
\end{array}$$

Arrows indicate the ordering that is true no matter from which direction we approach the point $(1,1)$. For instance, $(1-k^2)^2\psi_1(\lambda) = o((1-k^2)\psi_1(\lambda))$ as $k, \lambda \to 1$. However, unless we specify a curve in the (k,λ) plane this is only a partial ordering. For instance, we do not know how to compare $(1-k^2)^2\psi_1(\lambda)$ and $(1-k^2)^3\psi_2(\lambda)$. Choosing a curve possessing a tangent at the point $(1,1)$ introduces the complete ordering into the above matrix. Rearranging terms in the expansions (19) or (20) according to this ordering converts it into the asymptotic expansion for the specified curve. If the slope of our curve at $(1,1)$ is strictly between 0 and $\pi/2$, then all the diagonals of the above matrix will be of the same asymptotic order and the order will be decreasing as we move away from the main diagonal.

Remark 3. The error bound (21) shows that expansions (19) and (20) are convergent for k, λ in Region I. Taking $N = \infty$ in (19) and (20) and summing up we obtain convergent double series representations

$$\Pi(\lambda,\nu,k) = \sqrt{\frac{\nu}{1+\nu}}(\nu+k^2)^{-1/2}\arctan(\lambda\sqrt{\nu}) + \sum_{j=0}^{\infty}\sum_{n=0}^{j}\frac{(1/2)_j(1-k^2)^j}{(-1)^j j!}d_{j,n+1}(\nu)\psi_n(\lambda), \tag{25}$$

$$\Pi(\lambda,\nu,k) = \int_0^\lambda \frac{[\nu/(1+\nu)]dt}{(1+\nu t^2)\sqrt{1+\mu t^2}} + \sum_{j=0}^{\infty}\sum_{n=1}^{j+1}\frac{(1/2)_j\lambda^{2j+1}(1-k^2)^j\nu^{j-n+1}}{(-1)^j j!(2j+1)(1+\nu)^{j-n+2}}{}_2F_1(n, j+1/2; j+3/2; \lambda^2), \tag{26}$$

where $\mu = \nu(1-k^2)/(1+\nu)$. The first series requires additional condition $-k^2 < \nu$ for convergence.

To derive an expansion valid in Region II we start with an expansion valid in a neighbourhood of $(0,0)$ given by Carlson [1]:

$$\Pi(\lambda,\nu,k) = \sum_{m=0}^{\infty}\frac{\lambda^{2m+1}}{2m+1}\sum_{n=0}^{m}(-\nu)^{m-n}\frac{(1/2)_n}{n!}{}_2F_1(-n, 1/2; 1/2-n; k^2). \tag{27}$$

Using

$$\frac{(1/2)_n}{n!}{}_2F_1(-n,1/2;1/2-n;k^2) = {}_2F_1(-n,1/2;1;1-k^2), \tag{28}$$

which is a limiting case of the well-known analytic extension formula for ${}_2F_1$ (see [4, formula 2.10(1)]), we obtain the expansion

$$\Pi(\lambda,\nu,k) = \sum_{m=0}^{\infty} \frac{\lambda^{2m+1}}{2m+1} \sum_{n=0}^{m} (-\nu)^{m-n} {}_2F_1(-n,1/2;1;1-k^2), \tag{29}$$

which is valid for $|\nu|\lambda^2 < 1$ and $|k| < 1/\lambda$ as shown by Carlson. An application of (18) leads to

Theorem 2 *For* $k,\lambda\in$Region II, ν *in the range*

$$((1-\lambda^2)|\nu|)/(\nu+1) < 1, \tag{30}$$

and positive integer N *the following expansion holds true*

$$\Pi(\lambda,\nu,k)=\Pi(\nu,k) - \sqrt{\frac{1-\lambda^2}{1-k^2}} \sum_{m=0}^{N-1} \frac{(1-\lambda^2)^m}{2m+1} \sum_{n=0}^{m} \frac{\nu^{m-n}{}_2F_1(-n,1/2;1;(1-k^2)^{-1})}{(1+\nu)^{m-n+1}} + R_{2,N}(k,\nu,\lambda), \tag{31}$$

with the error bound given by

$$|R_{2,N}(k,\nu,\lambda)| \le \frac{(1-\lambda^2)^{N+1/2}}{2N+1} \left[\max\left(\frac{k^2}{1-k^2}, \frac{|\nu|}{\nu+1}, 1\right)\right]^{N+1/2} f(\lambda,k,\nu), \tag{32}$$

where $f(\lambda,k,\nu)$ *is defined by*

$$f(\lambda,k,\nu) = \begin{cases} \frac{1}{(1+\nu)k}\left[1-\frac{(1-\lambda^2)k^2}{1-k^2}\right]^{-1}\left[1-\frac{|\nu|(1-k^2)}{(\nu+1)k^2}\right]^{-1}, & \max\left(\frac{k^2}{1-k^2},\frac{|\nu|}{\nu+1}\right) = \frac{k^2}{1-k^2} > 1; \\ \frac{1}{\sqrt{(1-k^2)(1+\nu)|\nu|}}\left[1-\frac{(1-\lambda^2)|\nu|}{1+\nu}\right]^{-1}\left[1-\frac{(\nu+1)k^2}{|\nu|(1-k^2)}\right]^{-1}, & \frac{|\nu|}{\nu+1} > \frac{k^2}{1-k^2} > 1; \\ \frac{1}{\sqrt{(1-k^2)(1+\nu)|\nu|}}\left[1-\frac{(1-\lambda^2)|\nu|}{1+\nu}\right]^{-1}\left(\left[1-\frac{1+\nu}{|\nu|}\right]^{-1}+\frac{1}{\lambda^2}\right), & \frac{|\nu|}{\nu+1} > 1 \ge \frac{k^2}{1-k^2}; \\ \frac{1}{(1+\nu)\sqrt{1-k^2}}\left[1-\frac{(1-\lambda^2)|\nu|}{1+\nu}\right]^{-1}\left(\left[1-\frac{|\nu|}{1+\nu}\right]^{-1}+\frac{1}{\lambda^2}\right), & \max\left(\frac{k^2}{1-k^2},\frac{|\nu|}{\nu+1},1\right) = 1; \\ \frac{1}{\sqrt{(1-k^2)(1+\nu)|\nu|}}\left[1-\frac{(1-\lambda^2)|\nu|}{(1+\nu)}\right]^{-1}\left(\left[1-\frac{(1-\lambda^2)|\nu|}{(1+\nu)}\right]^{-1}+N\right), & \max\left(\frac{k^2}{1-k^2},1\right) = \frac{|\nu|}{\nu+1}. \end{cases} \tag{33}$$

Proof. To derive the form of (31) expand the second term on the righthand side of (18) into series (29). Carlson's conditions $|\nu|\lambda^2 < 1$ and $|k| < 1/\lambda$ written for the parameters of this second term become (3) and (30). Now consider the remainder term

$$R_N(k,\nu,\lambda) = \frac{1}{1+\nu}\left[\frac{1-\lambda^2}{1-k^2}\right]^{1/2} \tilde{R}_N(k,\nu,\lambda),$$

$$\tilde{R}_N(k,\nu,\lambda) = \sum_{m=N}^{\infty} \frac{(1-\lambda^2)^m}{2m+1} \sum_{n=0}^{m} \left(\frac{\nu}{1+\nu}\right)^{m-n} {}_2F_1(-n,1/2;1;(1-k^2)^{-1}).$$

Change the order of summation to get

$$\begin{aligned} \tilde{R}_N(k,\nu,\lambda) &= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} \frac{(1-\lambda^2)^{s+\max(n,N)}}{2(s+\max(n,N))+1}\left(\frac{\nu}{1+\nu}\right)^{s+\max(n,N)-n} {}_2F_1(-n,1/2;1;(1-k^2)^{-1}) \\ &= \sum_{n=0}^{\infty}(1-\lambda^2)^n {}_2F_1(-n,1/2;1;(1-k^2)^{-1})\left[\frac{(1-\lambda^2)\nu}{1+\nu}\right]^{\max(0,N-n)} \sum_{s=0}^{\infty}\frac{(1-\lambda^2)^s(\nu/(1+\nu))^s}{2(s+\max(n,N))+1}. \end{aligned} \tag{34}$$

Since $(L > -1/2)$

$$\sum_{s=0}^{\infty} \frac{x^s}{2s+2L+1} = \frac{x^{-L-1/2}}{2} \int_0^x \frac{t^{L-1/2}}{1-t} dt,$$

we may write

$$\tilde{R}_N(k,\nu,\lambda) = \frac{1}{2}\left[\frac{1+\nu}{(1-\lambda^2)\nu}\right]^{1/2} \int_0^{\frac{(1-\lambda^2)\nu}{1+\nu}} \frac{dt}{\sqrt{t}(1-t)} \sum_{n=0}^{\infty} \left[\frac{1+\nu}{\nu}\right]^n t^{\max(n,N)} {}_2F_1(-n,1/2;1;(1-k^2)^{-1}). \tag{35}$$

The series in (35) can be broken into two parts:

$$t^N \sum_{n=0}^{N-1} \left[\frac{1+\nu}{\nu}\right]^n {}_2F_1(-n,1/2;1;(1-k^2)^{-1}) + \sum_{n=N}^{\infty} \left[\frac{1+\nu}{\nu}\right]^n t^n {}_2F_1(-n,1/2;1;(1-k^2)^{-1}). \tag{36}$$

To give a bound for (35) we will need two ingredients. First is the estimate

$$|{}_2F_1(-n,1/2;1;(1-k^2)^{-1})| \le \begin{cases} |k^2/(1-k^2)|^n, & 1/2 \le k^2 < 1, \\ 1, & 0 \le k^2 \le 1/2, \end{cases} \tag{37}$$

which is obtained by writing ${}_2F_1$ as the Legendre polynomial [10, formula(7.3.176)]:

$${}_2F_1(-n,1/2;1;(1-k^2)^{-1}) = \left[\frac{-k^2}{1-k^2}\right]^{\frac{n}{2}} P_n\left(\frac{1-2k^2}{2\sqrt{-k^2(1-k^2)}}\right) = i^n \left[\frac{k^2}{1-k^2}\right]^{\frac{n}{2}} P_n\left(\frac{-i(1-2k^2)}{2k\sqrt{(1-k^2)}}\right) \tag{38}$$

and applying the inequality (valid for all complex z) $|P_n(z)| \le |z+\sqrt{z^2-1}|^n$, where the branch of the square root is chosen so that $|z+\sqrt{z^2-1}| \ge 1$. This inequality follows immediately from the first Laplace integral for $P_n(z)$ (see [4, formula 3.7(6)]). The second ingredient is the asymptotically precise (as $x \to 0$) estimate (verified by comparing derivatives of both sides and noticing that $f(0) = g(0)$):

$$f(x) = \int_0^x \frac{t^M dt}{1-t} \le \frac{x^{M+1}}{(M+1)(1-x)} = g(x), \tag{39}$$

valid for $M > -1$ and $0 \le x < 1$. To prove (32)–(33) we need to consider the following cases:
a) $1/2 \le k^2 < 1$, $(1+\nu)k^2/(|\nu|(1-k^2)) > 1$; b) $1/2 \le k^2 < 1$, $(1+\nu)k^2/(|\nu|(1-k^2)) < 1$;
c) $1/2 \le k^2 < 1$, $(1+\nu)k^2/(|\nu|(1-k^2)) = 1$; d) $0 \le k^2 \le 1/2$, $\nu > -1/2$;
e) $0 \le k^2 \le 1/2$, $-1 < \nu < -1/2$; and f) $0 \le k^2 \le 1/2$, $\nu = -1/2$.
Take a). Inequality (37) gives for the first sum in (36):

$$|t|^N \sum_{n=0}^{N-1} \left|\frac{1+\nu}{\nu}\right|^n |{}_2F_1(-n,1/2;1;(1-k^2)^{-1})| \le \left|\frac{t(1+\nu)k^2}{\nu(1-k^2)}\right|^N \frac{1}{[(\nu+1)k^2]/[|\nu|(1-k^2)]-1}.$$

Similarly, the second sum in (36) satisfies

$$\left|\sum_{n=N}^{\infty} \left[\frac{1+\nu}{\nu}\right]^n t^n {}_2F_1(-n,1/2;1;(1-k^2)^{-1})\right| \le \frac{\left|\frac{t(1+\nu)k^2}{\nu(1-k^2)}\right|^N}{1-\left|\frac{t(1+\nu)k^2}{\nu(1-k^2)}\right|} \le \frac{\left|\frac{t(1+\nu)k^2}{\nu(1-k^2)}\right|^N}{1-\frac{(1-\lambda^2)k^2}{(1-k^2)}}.$$

The last inequality is due to the fact that $t \in [0,(1-\lambda^2)|\nu|/(\nu+1)]$. Hence the complete sum (36) is estimated by

$$\left|\sum_{n=0}^{\infty} \left[\frac{1+\nu}{\nu}\right]^n t^{\max(n,N+1)} {}_2F_1(-n,1/2;1;(1-k^2)^{-1})\right| \le \left|\frac{t(1+\nu)k^2}{\nu(1-k^2)}\right|^N \frac{\frac{k^2(1-\lambda^2)}{(1-k^2)}\left[\frac{\nu+1}{|\nu|(1-\lambda^2)}-1\right]}{\left[\frac{(\nu+1)k^2}{|\nu|(1-k^2)}-1\right]\left[1-\frac{(1-\lambda^2)k^2}{1-k^2}\right]}.$$

Substituting this estimate into (35) and using (39) we get:

$$|\tilde{R}_N(k,\nu,\lambda)| \le \frac{1+\nu}{(2N+1)|\nu|}\left[\frac{(1-\lambda^2)k^2}{(1-k^2)}\right]^N \frac{k^2/(1-k^2)}{\left[\frac{(\nu+1)k^2}{|\nu|(1-k^2)}-1\right]\left[1-\frac{(1-\lambda^2)k^2}{1-k^2}\right]},$$

which is equivalent to (32) plus first line of (33). Proofs in all other cases require only minor modifications. $\square$

Remark 4. Error bound (32) shows that expansion (31) is convergent and asymptotic as $k\to 1$ with either constant λ or $\lambda\to 1$ in a way that $(1-k)/(1-\lambda)\to 0$, so that the point $(1,1)$ is approached along a curve in (k,λ) plane having zero slope at $(1,1)$. It can be rearranged into an asymptotic expansion for $k,\lambda\to 1$ along an arbitrary curve as explained in Remark 2.

Remark 5. Expansion (31) can be derived without any use of Carlson's expansion (27) by expanding the rightmost term in (18) into binomial series, similarly to the proof of Theorem 1.

Remark 6. The complete elliptic integral $\Pi(\nu,k)$ in (31) can be approximated using the following asymptotic expansion valid for $k\to 1$:

$$\Pi(\nu,k)=\sum_{n=0}^{M-1}\frac{(1/2)_n(1/2)_n}{(n!)^2}(1-k^2)^n(\psi(1+n)-\psi(1/2+n)-\frac{1}{2}\ln(1-k^2))_2F_1(1,1/2+n;1/2;-\nu)$$
$$-\sum_{n=0}^{M-1}\frac{(1/2)_n(1/2)_n}{2(n!)^2}(1-k^2)^n{}_2F_1{}^{(0,1,0,0)}(1,1/2+n;1/2;-\nu)+\mathcal{O}\left((1-k)^M\ln\frac{1}{1-k}\right). \quad (40)$$

Here ${}_2F_1{}^{(0,1,0,0)}$ is the derivative of ${}_2F_1$ in the second parameter. This expansion can be obtained by writing $\Pi(\nu,k)$ as Appell's hypergeometric function F_1 (see [1]), expressing F_1 as a series in terms of Gauss hypergeometric functions ${}_2F_1$ and applying [4, formula 2.10(12)] to ${}_2F_1$.

4. Results of computations.

In this section we give several examples of computations with the expansions obtained above. Denote by $\widetilde{\Pi}_N(\lambda,\nu,k)$ the N-th order approximation in (19), i.e. the right hand side of (19) without the remainder term $R_{1,N}(\lambda,\nu,k)$. Then:

$$\widetilde{\Pi}_1(\lambda,\nu,k)=\frac{1}{2(1+\nu)}\ln\frac{1+\lambda}{1-\lambda}+\frac{\sqrt{\nu}\arctan(\lambda\sqrt{\nu})}{1+\nu},$$

$$\widetilde{\Pi}_2(\lambda,\nu,k)=\widetilde{\Pi}_1(\lambda,\nu,k)-\frac{1-k^2}{8(1+\nu)(1-\lambda)}\left\{1+(1-\lambda)\frac{\nu-1}{\nu+1}\ln\frac{1+\lambda}{1-\lambda}\right.$$
$$\left.-4\frac{1-\lambda}{1+\nu}\sqrt{\nu}\arctan(\lambda\sqrt{\nu})-\frac{1-\lambda}{1+\lambda}\right\},$$

$$\widetilde{\Pi}_3(\lambda,\nu,k)=\widetilde{\Pi}_2(\lambda,\nu,k)+\frac{3}{128(1+\nu)}\frac{(1-k^2)^2}{(1-\lambda)^2}\left\{1-2\lambda\left(\frac{4}{1+\nu}+1\right)\frac{1-\lambda}{1+\lambda}\right.$$
$$\left.+\ln\frac{1+\lambda}{1-\lambda}\left(\frac{8}{(1+\nu)^2}-\frac{4}{1+\nu}-1\right)(1-\lambda)^2+16\frac{(1-\lambda)^2\sqrt{\nu}\arctan(\lambda\sqrt{\nu})}{(1+\nu)^2}-\frac{(1-\lambda)^2}{(1+\lambda)^2}\right\},$$

λ	k	$\frac{1-k}{1-\lambda}$	$\Pi(\nu,\lambda,k)$	$\widetilde{\Pi}_1(\nu,\lambda,k)$	relative error	relative error bound	$\widetilde{\Pi}_3(\lambda,\nu,k)$	relative error	relative error bound
.5	.9	.2	.37138	.37409	$-.730\times10^{-2}$	$.213\times10^{-1}$	.37138	$-.634\times10^{-5}$	$.178\times10^{-4}$
.6	.99	.025	.41973	.42022	$-.117\times10^{-2}$	$.400\times10^{-2}$	.41973	$-.306\times10^{-7}$	$.104\times10^{-6}$
.75	.999	.004	.48662	.48673	$-.218\times10^{-3}$	$.990\times10^{-3}$	.48662	$-.289\times10^{-9}$	$.136\times10^{-8}$
.9	.99999	.0001	.57202	.57203	$-.553\times10^{-5}$	$.335\times10^{-4}$	.57202	$-.784\times10^{-14}$	$.508\times10^{-13}$

Table 4.1. *Numerical examples for the approximation* (19) *with* $k\to 1$, $\lambda\to 1$, $(1-k)/(1-\lambda)\to 0$ *and* $\nu=7$. *The sixth and the ninth columns represents the relative errors* $R_{1,N}(\lambda,\nu,k)/\Pi(\lambda,\nu,k)$ *in*

(19). *The seventh and the tenths columns represent relative error bounds as given by the rhs of* (21) *divided by* $\Pi(\lambda,\nu,k)$.

For k and λ satisfying (3) denote by $\widehat{\Pi}_N(\lambda,\nu,k)$ the N-th order approximation from (31), i.e. the right hand side of (31) without remainder term $R_{2,N}(\lambda,\nu,k)$. Then:

$$\widehat{\Pi}_1(\lambda,\nu,k)=\Pi(\nu,k)-\frac{1}{1+\nu}\left(\frac{1-\lambda^2}{1-k^2}\right)^{1/2},$$

$$\widehat{\Pi}_2(\lambda,\nu,k)=\widehat{\Pi}_1(\lambda,\nu,k)-\frac{1}{3}\left(\frac{1-\lambda^2}{1-k^2}\right)^{3/2}\left\{\frac{1-k^2}{1+\nu}\left(1+\frac{\nu}{1+\nu}\right)-\frac{1}{2}\frac{1}{1+\nu}\right\},$$

$$\widehat{\Pi}_3(\lambda,\nu,k)=\widehat{\Pi}_2(\lambda,\nu,k)-\frac{1}{5}\left(\frac{1-\lambda^2}{1-k^2}\right)^{5/2}\left\{\frac{(1-k^2)^2}{1+\nu}\left(1+\frac{\nu}{1+\nu}+\frac{\nu^2}{(1+\nu)^2}\right)\right.$$
$$\left.-\frac{1}{2}\frac{1-k^2}{1+\nu}\left(2+\frac{\nu}{1+\nu}\right)+\frac{3}{8}\frac{1}{1+\nu}\right\},$$

λ	k	$\frac{1-\lambda}{1-k}$	$\Pi(\lambda,\nu,k)$	$\widehat{\Pi}_1(\lambda,\nu,k)$	relative error	relative error bound	$\widehat{\Pi}_3(\lambda,\nu,k)$	relative error	relative error bound
.9	.5	.2	.50760	.51315	$-.109\times10^{-1}$	$.867\times10^{-1}$	.50770	$-.212\times10^{-3}$	$.135\times10^{-2}$
.99	.6	.025	.56514	.56530	$-.287\times10^{-3}$	$.238\times10^{-2}$	.56514	$-.606\times10^{-7}$	$.403\times10^{-6}$
.999	.75	.004	.60555	.60556	$-.682\times10^{-5}$	$.376\times10^{-4}$	.60555	$-.138\times10^{-10}$	$.106\times10^{-9}$
.9999	.8	.0005	.62453	.62453	$-.153\times10^{-6}$	$.110\times10^{-5}$	.62453	$-.888\times10^{-15}$	$.597\times10^{-13}$
.999999	.95	.00002	.71429	.71429	$.172\times10^{-8}$	$.540\times10^{-8}$	.71429	$.190\times10^{-18}$	$.793\times10^{-18}$

Table 4.2. *Numerical examples for the approximation* (31) *with* $\lambda\to1$, $k\to1$, $(1-\lambda)/(1-k)\to0$ *and* $\nu=7$. *The sixth and the ninth columns represents the relative errors* $R_{2,N}(\lambda,\nu,k)/\Pi(\lambda,\nu,k)$ *in* (31). *The seventh and the tenths columns represent relative error bounds as given by the rhs of* (32)-(33) *divided by* $\Pi(\lambda,\nu,k)$.

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